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Matching quantum strings to quantum spins: one-loop vs. finite-size corrections

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Abstract

We compare quantum corrections to semiclassical spinning strings in $AdS_5 \times S^5$ to one-loop anomalous dimensions in $\mathcal{N} = 4$ supersymmetric gauge theory. The latter are computed using the reduced (Landau-Lifshitz) sigma model and with the help of the Bethe ansatz. The results of all three approaches are in remarkable agreement with each other. As a byproduct we establish the relationship between linear instabilities in the Landau-Lifshitz model and analyticity properties of the Bethe ansatz.

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1 Introduction

Recently, there was a remarkable progress towards uncovering the structure of the spectrum of energies of (non-interacting) quantum strings in $AdS_5 \times S^5$ or, equivalently, the spectrum of dimensions of single-trace operators in the dual $\mathcal{N} = 4$ SYM theory with $N \to \infty$, $\lambda = g^2_{\rm YM} N$ =fixed (for reviews see, e.g., [1, 2, 3, 4]). Both energies E and dimensions Δ depend on the 't Hooft coupling λ (the string tension is $T = \frac{\sqrt{\lambda}}{2\pi}$) as well as on quantum numbers like spins J and "winding" numbers m characterizing the states, and one basic implication of the AdS/CFT duality is the equality of the functions $E(\lambda, J, m, ...) = \Delta(\lambda, J, m, ...)$ for *any* value of the arguments. It is not a priori clear how such a relation can be tested for far-from-BPS states, but remarkably it was found that both the perturbative string theory and the perturbative gauge theory (where anomalous dimensions are described by a spin chain) contain certain states for which the large J expansions of E and Δ have similar structures (see, e.g., [5, 6, 7] and [8, 9, 10, 11, 12, 13])[1]

$$E = J\Big[1 + \frac{\lambda}{J^2}(c_0^{(1)} + \frac{c_1^{(1)}}{J} + \frac{c_2^{(1)}}{J^2} + ...) + \frac{\lambda^2}{J^4}(c_0^{(2)} + \frac{c_1^{(2)}}{J} + \frac{c_2^{(2)}}{J^2} + ...) + ...\Big] \, , \qquad (1.1)$$

$$\Delta = J\Big[1 + \frac{\lambda}{J^2}(a_1^{(0)} + \frac{a_1^{(1)}}{J} + \frac{a_1^{(2)}}{J^2} + ...) + \frac{\lambda^2}{J^4}(a_2^{(0)} + \frac{a_2^{(1)}}{J} + \frac{a_2^{(2)}}{J^2} + ...) + ...\Big] \, . \qquad (1.2)$$

The coefficients $c_\ell^{(n)}$ (corresponding to ℓ-loop string-theory correction $\sim (\frac{1}{\sqrt{\lambda}})^{\ell+1}$, $J = \sqrt{\lambda}\mathcal{J}$) and $a_\ell^{(n)}$ (corresponding to ℓ-loop gauge-theory correction $\sim \lambda^\ell$) depend on ratios of spins and other quantum numbers and are finite in the large J limit. Similar expressions are found for near-BPS states describing small string fluctuations near the BMN vacuum state.[2]

The two expressions, however, are obtained in the two different limits. On the string side one uses semiclassical expansion in which $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ is kept fixed while one first expands in $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ or, equivalently, in $\frac{1}{J}$; one may then also expand in $\tilde{\lambda} = \frac{1}{\mathcal{J}^2} = \frac{\lambda}{J^2}$, which corresponds to studying "fast-moving" strings. On gauge-theory side, one uses standard planar perturbation theory, i.e. first expands in λ and then may also expand the resulting l-loop anomalous dimensions in large J (or, equivalently, for the simplest cases we are going to consider) in large length of the operator. What is even more surprising (given that the string and gauge theory limits are "opposite" as far as λ is concerned while $E = \Delta$ should in general contain non-trivial interpolating functions

[1] Remarkably, the spectrum of the corresponding ferromagnetic spin chains does contain "macroscopic string" states [14, 8] for which the spin chain energies or anomalous dimensions scale as $\frac{\lambda^\ell}{J^{2\ell-1}}$ at $\ell = 0, 1, 2, ...$ loop orders.

[2] In this case [15, 16] one is to extract one power of J, i.e. $E = J + \frac{\lambda}{J^2}(c_1^{(1)} + \frac{c_2^{(1)}}{J} + \frac{c_3^{(1)}}{J^2} + ...) + \frac{\lambda^2}{J^4}(c_1^{(2)} + \frac{c_2^{(2)}}{J} + \frac{c_3^{(2)}}{J^2} + ...) + ...$, etc.

of λ) is that the first two leading coefficient functions (1.1) and (1.2) were found to be exactly the same:

$$c_0^{(1)} = a_1^{(0)} \ , \qquad c_0^{(2)} = a_2^{(0)} \ , \tag{1.3}$$

i.e. the two leading classical string theory coefficients match the two leading (one- and two-loop) gauge theory coefficients.

This matching can be demonstrated in a very general way by either extracting the corresponding coherent state ("Landau-Lifshitz") sigma model describing low-energy states of the ferromagnetic spin chain and matching it to a "fast-string" limit of the classical superstring action [17, 18, 19, 20, 21, 22, 23, 24], or by matching the integral equation for the general solutions of the integrable string sigma model to a similar equation for the density of the Bethe root distribution appearing on the spin chain side [25, 26, 27, 28, 29]. This matching applies also to near-by fluctuations, and is of a novel truly "microscopic" or "dynamical" nature, i.e. it appears to go beyond of a kind of matching based on a non-renormalization theorem or BPS-saturation of a coefficient of a certain term in a gauge or supergravity effective action familiar in other (e.g, in matrix theory) contexts.

An explanation of why one gets this agreement for the two leading coefficients but apparently $c_0^{(n)} \neq a_n^{(0)}$, $n > 2$ [11, 30] (so that one needs to resum the series to verify that $E = \Delta$) may be traced to the structure of the dilatation operator (or 1-d S-matrix [31]) on the gauge theory side. The structure of the dilatation operator (best understood so far in $SU(2)$ [32, 33, 30] or $SU(2|3)$ [34] sectors) is dictated to a large extent by the maximal supersymmetry. The observed matchings (for near-BMN states to two orders in $\frac{\lambda}{J^2}$ and first order in $\frac{1}{J}$; for spinning string states to two leading orders in $\frac{\lambda}{J^2}$) can be attributed to the fact that the the "gauge-theory" [33] and the "string-theory" [35, 36] Bethe ansatze start to disagree at λ^3 order, while both should be limits of a "Better ansatz" [35] that contains interpolating functions of λ and J.

This suggests that one should expect more matching between the coefficients in (1.1) and (1.2) at the *first two* orders in λ, in particular,

$$c_1^{(1)} = a_1^{(1)} \ , \tag{1.4}$$

i.e. the *1-loop* quantum string theory correction to the semiclassical string energy should match the leading finite-size correction to the *1-loop* gauge-theory anomalous dimension. This is a novel situation since (like near-BMN example) it involves quantum string theory result (incorporating, in particular, fermionic contributions) while previous tests where for purely bosonic classical string solutions.[3]

Attempts to test (1.4) were made previously in [38, 39] where string 1-loop corrections to energies of particular circular strings were computed and were compared to the spin

[3]The possibility of matching of the leading classical coefficients (1.3) does depend implicitly on the full structure of the quantum superstring theory: this matching depends on the fact that quantum string corrections are suppressed in the large J limit, which itself is a consequence of the 2-d conformal invariance and underlying supersymmetry of the superstring sigma model [5, 37].

chain results for the $1/J$ corrections found in [40, 26]. An apparent disagreement was reported: it appeared that only the zero-mode string theory result for the 1-loop correction to the energy (given by a familiar sum of the fluctuation mode frequencies) was captured by the Bethe ansatz [40, 26]. We shall compute an additional anomalous contribution to the Bethe ansatz[4] overlooked in [40, 26], see Appendix A, which restores the agreement with the string calculation.

In section 2 we shall review the 1-loop string results of [37, 38, 39] for the leading $1/J$ correction to the energy of circular spinning strings. We shall explain in particular that the full superstring 1-loop correction to the string energy can be understood as a ζ-function regularized expression for the 1-loop correction to the string soliton energy computed directly in the corresponding "reduced" or "Landau-Lifshitz" sigma-model (i.e. in the continuum limit of the coherent state path integral corresponding to the spin chain Hamiltonian)[5]. Then in section 3 we shall show how a careful account of an anomaly (discussed in Appendix A) appearing in the finite-size expansion of the Bethe ansatz equations leads to the same expression as found on the string side.

Given that the string expression is essentially the regularized result of the Landau-Lifshitz model and that the energies of the small-fluctuation Landau-Lifshitz modes can be reproduced [8] on the Bethe ansatz side, this may seem to make the agreement quite natural (modulo the fact that the two computations still apply in two different limits). This stresses again a "microscopic" nature of the matching: not only the final expressions for the coefficients match but also there is a remarkable correspondence between the intermediate steps in the respective calculations. Surprisingly, the Landau-Lifshitz model continues to provide a conceptual link between the string theory and the spin chain even beyond the leading semiclassical approximation.

Let us mention also that our result may shed more light on the AFS [35] ansatz which is *a* "discretization" of the classical string sigma model solution. As was shown in [36], the AFS ansatz gives rise to a spin chain at small coupling λ. This spin chain agrees precisely, including all $1/J$ effects, with gauge theory up to two loops. The present work gives support to the idea that the quantum string at small λ is indeed described by a spin chain, be it the one of gauge theory [32, 33] or the string chain of [36] (which are both equivalent to the Heisenberg spin chain at this order).

[4] We would like to thank V. Kazakov for inspiring discussions on possible anomalies in the Bethe equations.

[5] The quantization of the classical solutions in the Landau-Lifshitz model with the help of zeta-regularization was considered before by J. Minahan (unpublished). We would like to thank him for the discussion of these results.

2 One-loop superstring vs. quantum Landau-Lifshitz model

2.1 $SU(2)$ case

It is best to start with the simplest possible two-spin solution in the $SU(2)$ sector [5]: rigid circular string rotating with two equal angular momenta $J_1 = J_2 = \frac{1}{2}J$ in S^3 part of S^5. In the form given in [7] (equivalent to the solution of [5] by an $SO(4)$ rotation) it is $t = \kappa\tau$, $\mathrm{X}_1 = \cos\psi\ e^{i\phi_1} = \frac{1}{\sqrt{2}}e^{iw\tau+ik\sigma}$, $\mathrm{X}_2 = \sin\psi\ e^{i\phi_2} = \frac{1}{\sqrt{2}}e^{iw\tau-ik\sigma}$. Here $|\mathrm{X}_1|^2 + |\mathrm{X}_2|^2 = 1$, $w = \mathcal{J} = \frac{J}{\sqrt{\lambda}}$, $\kappa^2 = \mathcal{J}^2 + k^2$ and k is an integer winding number. The classical energy $E = \sqrt{\lambda}\kappa$ of this solution is [5] (cf. (1.1))

$$E = \sqrt{J^2 + \lambda k^2} = J(1 + \frac{\lambda k^2}{2J^2} + ...) \ . \tag{2.1}$$

To find the string 1-loop correction to E one is supposed to determine the characteristic frequencies ω_n of the bosonic and fermionic fluctuations $\sim e^{i\omega_n\tau+in\sigma}$ and then compute an appropriate sum over them. Considering the AdS_5 time t and the "fast" motion direction $\frac{1}{2}(\phi_1 + \phi_2)$ as the "longitudinal" directions, there will be 8 bosonic and 8 fermionic fluctuations. Among the bosonic ones, the remaining two of S^3 fluctuations (ψ and $\frac{1}{2}(\phi_1 - \phi_2)$) play a special role compared to four AdS_5 and two other S^5 fluctuations): they belong to the $SU(2)$ Landau-Lifshitz sigma-model [17, 18]. Using the results of [37] (summarized in Appendix B of [38]) we then find the expression for the 1-loop correction as a sum of the zero ($n = 0$ or constant in σ) mode and non-zero mode contributions[6]

$$E_1 = E_{\rm zero} + E_{\rm non-zero} \ , \qquad\qquad E_{1\ \rm non-zero} = \sum_{n=1}^{\infty} S_n \ , \tag{2.2}$$

$$E_{\rm zero} = 2 + \sqrt{1 - \frac{2k^2}{\mathcal{J}^2 + k^2}} - 3\sqrt{1 - \frac{k^2}{\mathcal{J}^2 + k^2}} \ , \tag{2.3}$$

$$\begin{aligned} S_n = \quad & 2\sqrt{1 + \frac{(n + \sqrt{n^2 - 4k^2})^2}{4(\mathcal{J}^2 + k^2)}} + 2\sqrt{1 + \frac{n^2 - 2k^2}{\mathcal{J}^2 + k^2}} + 4\sqrt{1 + \frac{n^2}{\mathcal{J}^2 + k^2}} \\ & -8\ \sqrt{1 + \frac{n^2 - k^2}{\mathcal{J}^2 + k^2}} \ . \end{aligned} \tag{2.4}$$

[6]We are using here that in the case of the "homogeneous" solutions of [7] the isometric angles are linear functions of τ and σ and so their derivatives (and thus all coefficients in the fluctuation Lagrangian) are constant. Also, the connection in the fermionic covariant derivative is automatically constant (one does not need a σ-dependent rotation considered in [37]), and thus the fermions are periodic and their modes are labelled by the integers n just like as the bosonic modes.

The sum over n is finite as a consequence of the conformal invariance of the string theory [41, 5, 37]: the bosonic and fermionic divergences cancel each other. The first ("fourth root") term in S_n in (2.4) is the contribution of the two "transverse" S^3 fluctuation modes. Note that the latter with $n < 2k$ are tachyonic [5] and thus contribute to an imaginary part of E_1. We shall ignore this problem as a very similar discussion will apply also in the stable $SL(2)$ case considered in [39] and below; moreover, we will still be able to formally match this string result to the $SU(2)$ spin chain one despite this instability problem.

Expanding the above expression at large $\mathcal{J}$ or small $\tilde{\lambda} = \frac{1}{\mathcal{J}^2} = \frac{\lambda}{J^2}$ we find:

$$E_{\rm zero} = \frac{\lambda k^2}{2J^2} + O(\frac{\lambda^2}{J^4}) \ , \tag{2.5}$$

$$E_{\rm non-zero} = \frac{\lambda}{2J^2} \sum_{n=1}^{\infty} \ (n\sqrt{n^2 - 4k^2} - n^2 + 2k^2) + O(\frac{\lambda^2}{J^4}) \ . \tag{2.6}$$

Note that the leading term in the zero-mode contribution (2.5) is exactly the same as in the classical energy (2.1) but with an extra $1/J$ factor. This appears to be a universal feature: we will find the same in the $SU(3)$ and $SL(2)$ sectors.

A justification for using the $1/\mathcal{J}$ expansion before doing the infinite sum is that the sum in (2.6) is again convergent.[7] Notice that the first $n\sqrt{n^2 - 4k^2}$ term in (2.6) originated from the large $\mathcal{J}$ expansion of the first term in the sum in (2.4) (which is equal to the sum of the two S^3 fluctuation frequencies) and is thus the contribution of the Landau-Lifshitz fluctuation mode. The other two terms $-n^2$ and $2k^2$ that make the sum in (2.6) finite receive contribution from all the bosonic *and* fermionic modes that thus conspire to make the sum finite.

Let us now compare this superstring result with what one finds using an apparently naive procedure based quantization of the Landau-Lifshitz sigma model. From the $SU(2)$ spin chain perspective, one first replaces the quantum mechanics of the Heisenberg ferromagnet by a coherent-state path integral with the action containing the coherent-state expectation value of the Hamiltonian, i.e. $< n|H|n >= \frac{\lambda}{16\pi^2} \sum_{a=1}^{J} (\vec{n}_{a+1} - \vec{n}_a)^2$, where $< n|\vec{\sigma}_a|n >= \vec{n}_a$. As discussed in [17, 18], concentrating on a particular class of low-energy coherent states one can then define a semiclassical limit of the coherent state path integral as $J \to \infty$ with $\tilde{\lambda} = \frac{\lambda}{J^2}$ fixed. Indeed, in this limit one is

[7] We have checked numerically (as in [38]) that the finite sum in (2.6) matches indeed the $1/\mathcal{J}^2$ coefficient in the expansion of the function obtained by computing first the sum in (2.2) (for $k = 1$ we got $E_{\rm zero} + E_{\rm non-zero} \approx \tilde{\lambda}(-0.4667 + 0.866i)$ as in [38]). In general, one may wonder why one can obtain the $1/\mathcal{J}$ coefficient by first expanding in $1/\mathcal{J}$ and then doing the sum over n since n can be bigger than $\mathcal{J}$. Indeed, the coefficients of the higher-order $1/\mathcal{J}^4$, etc. terms in the expansion are given by divergent series. What happens is that a resummation of the divergent part of the $1/\mathcal{J}$ expansion that makes the result finite (as it was originally in (2.2) before expanding frequencies in $1/\mathcal{J}$) should not change the coefficient of the leading finite $1/\mathcal{J}^2$ term.

able to take the continuum limit of the action,[8] ending up with the Landau-Lifshitz action for a unit 3-vector 2d field $\vec{n}(\tau,\sigma)$

$$I = J\int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi}\ [\ \vec{C}(n)\cdot\dot{\vec{n}}\ -\ \frac{1}{8}\ \tilde{\lambda}\ \vec{n}'\cdot\vec{n}'\ ]\ . \tag{2.7}$$

This action happens to be the same [17] as the "fast-motion" ($\tilde{\lambda}\to 0$) limit of the relevant $R_t\times S^3$ bosonic part of the $AdS_5\times S^5$ string action, demonstrating, in particular, the matching of the leading-order coefficients in (1.3). To go beyond the leading classical approximation would mean to include the leading $1/J$, i.e. one-loop, correction to the coherent state path integral.

On the spin chain side, given that we have first taken the continuum limit and *then* want to include quantum corrections, it is not a priori clear that quantization of the Landau-Lifshitz model is going to reproduce the finite-size or $1/J$ expansion of the Bethe ansatz solution. On the string side, first taking the "fast string" limit and then quantizing the resulting reduced bosonic sigma model (thus ignoring quantum fluctuations in other directions) appears bound to be wrong, since, in particular, one needs the fermionic contributions to make the 1-loop correction finite.[9]

And yet, this naive procedure manages to reproduce essentially the right answer for the leading λ/J^2 correction: it just needs to be supplemented by a specific regularization prescription dictated by the underlying microscopic theory (spin chain described by Bethe ansatz or quantum superstring). It turns out that the Bethe ansatz result dictates that this regularization should be the standard $e^{-\epsilon n}$ or ζ-function regularization. Moreover, this prescription happens to produce exactly the same result as the full superstring calculation – the role of other superstring modes happens to reduce just to making the sum of the Landau-Lifshitz frequencies finite!

To see how this happens in some detail, let us start with the Landau-Lifshitz equation on $R_\tau\times S^1_\sigma$ which follows from (2.7)

$$\dot{n}^i = \frac{1}{2}\tilde{\lambda}\ \epsilon^{ijk} n_j n_k''\ , \qquad \vec{n}^2 = 1\ , \qquad \tilde{\lambda}\equiv\frac{\lambda}{J^2}\ , \tag{2.8}$$

and consider the simplest non-trivial static solution corresponding to the circular string with $J_1 = J_2 = \frac{1}{2}J$

$$n_i = (\cos 2k\sigma,\ \sin 2k\sigma,\ 0)\ , \qquad k = 1,2,...\ . \tag{2.9}$$

Expanding (2.8) near this solution with small perturbations parametrized by two independent functions A_1, A_2

$$\delta n_i = (-\sin 2k\sigma\ A_1(\tau,\sigma),\ \cos 2k\sigma\ A_1(\tau,\sigma),\ A_2(\tau,\sigma))\ , \quad n_i\delta n_i = 0\ , \tag{2.10}$$

[8] Note that it is necessary to take the continuum limit in order to define the semiclassical expansion: only then the factor of J appears in front of the action and thus plays the role of the inverse Planck's constant.

[9] Modulo the UV divergence problem, one could try to justify this using the effective field theory philosophy observing that other fluctuations should be heavy in the large $\mathcal{J}$ limit.

it is easy to show that $\dot{A}_1 = -\frac{1}{2}\tilde{\lambda}(A_2'' + 4k^2 A_2)$, $\dot{A}_2 = \frac{1}{2}\tilde{\lambda} A_1''$. Expanding the fluctuations in modes $A_s \sim \sum_{n=-\infty}^{\infty} C_{s,n} e^{iw_n\tau + in\sigma}$ we find that the characteristic frequencies are given by

$$w_n = \pm\frac{1}{2}\tilde{\lambda} n\sqrt{n^2 - 4k^2} \ . \tag{2.11}$$

These fluctuation energies were indeed reproduced (for $n > 1$) also from the Bethe ansatz in [8].

As in the standard quantum oscillator case, the correction to the classical energy (cf. (2.1); superscript $^{(1)}$ indicates that this is order $\tilde{\lambda}$ contribution)

$$E_0^{(1)} = \frac{1}{8}\tilde{\lambda} J \int_0^{2\pi} \frac{d\sigma}{2\pi} n_i' n_i' = \frac{1}{2}\tilde{\lambda} J k^2 = \frac{\lambda k^2}{2J} \tag{2.12}$$

should then be given by

$$E_1^{(1)} = \frac{1}{2}\sum_{n=-\infty}^{\infty} |w_n| = \frac{1}{2}\tilde{\lambda}\sum_{n=1}^{\infty} n\sqrt{n^2 - 4k^2} \ . \tag{2.13}$$

This sum is divergent, but let us compute it by first adding and subtracting the divergent part and then renormalizing the divergent part using the $e^{-\epsilon n}$ regularization and dropping terms singular in the $\epsilon \to 0$ limit (this is equivalent to the ζ-function regularization prescription). We get

$$E_1^{(1)} = E_{\rm reg} + E_{\rm fin} \ , \qquad E_{\rm reg} = \frac{\lambda}{2J^2}[\sum_{n=1}^{\infty}(n^2 - 2k^2)]_{\rm reg} \ , \tag{2.14}$$

$$E_{\rm fin} = \frac{\lambda}{2J^2}\sum_{n=1}^{\infty}(n\sqrt{n^2 - 4k^2} - n^2 + 2k^2) \ . \tag{2.15}$$

The finite part of the regularization-dependent term $E_{\rm reg}^{(1)}$ is then[10]

$$E_{\rm reg} = \frac{1}{2}\tilde{\lambda} k^2 = \frac{\lambda k^2}{2J^2} \ . \tag{2.16}$$

Comparing (2.16) and (2.15) with the string results (2.5) and (2.6) we conclude that

(i) the regularized value $E_{\rm reg}$ of the divergent part of the sum of the Landau-Lifshitz fluctuation modes happens to be the same as the leading zero-mode contribution, and

(ii) the finite sum of *all* non-zero mode string contributions turns out to be equal just to the finite part $E_{\rm fin}$ of the sum of the Landau-Lifshitz modes.

The string theory thus provides an automatic regularization of the Landau-Lifshitz mode contributions. Given that the Landau-Lifshitz fluctuations are "visible" [8, 42]

[10] We use that $[\sum_{n=1}^{\infty} n^2]_{\rm reg} = \zeta(-2) = 0$, $[\sum_{n=1}^{\infty} 1]_{\rm reg} = \zeta(0) = -\frac{1}{2}$. Note also that the divergence coming from the n^2 term could be ruled out by the condition that the result should vanish in the case of $k = 0$ which corresponds to fluctuations near the BMN vacuum $\vec{n} = (1, 0, 0)$.

on the spin chain side this gives a strong hint that there should be a precise matching between the quantum string and the spin chain results for the $1/J$ correction. As we shall explain in section 3 below, the one-loop anomalous dimension computed from the Bethe ansatz equations indeed agrees with the string calculation.

The above expressions can be readily generalized to the case of the $SU(2)$ circular solution with two unequal spins [7]: $\mathrm{X}_1 = \cos\psi_0\ e^{iw_1\tau + ik_1\sigma}$, $\mathrm{X}_2 = \sin\psi_0\ e^{iw_2\tau - ik_2\sigma}$, where $k_1 J_1 = k_2 J_2,\ J = J_1 + J_2$ and the classical energy has the expansion (we assume $k_1, k_2 > 0$)

$$E = J + \frac{\lambda}{2J^2}(k_1^2 J_1 + k_2^2 J_2) + O(\frac{\lambda^2}{J^3}) = J + \frac{\lambda}{2J^2} M^2 + O(\frac{\lambda^2}{J^3})\ , \qquad (2.17)$$

$$M^2 \equiv m^2 \alpha(1-\alpha)\ , \qquad m \equiv k_1 + k_2\ , \qquad \alpha \equiv \frac{J_2}{J}\ . \qquad (2.18)$$

The above equal-spin case of $k_1 = k_2 = k,\ J_1 = J_2$ corresponds to $\alpha = \frac{1}{2},\ m = 2k,\ M = k$. The generalizations of (2.5),(2.16) and (2.6),(2.15) are found to be

$$E_{\rm zero} = E_{\rm reg} + O(\frac{\lambda^2}{J^4})\ , \qquad E_{\rm reg} = \frac{\lambda}{2J^2}[\sum_{n=1}^{\infty}(n^2 - 2M^2)]_{\rm reg} = \frac{\lambda}{2J^2} M^2\ , \qquad (2.19)$$

$$E_{\rm non-zero} = E_{\rm fin} + O(\frac{\lambda^2}{J^4})\ , \qquad E_{\rm fin} = \frac{\lambda}{2J^2}\sum_{n=1}^{\infty}(n\sqrt{n^2 - 4M^2} - n^2 + 2M^2)\ . \quad (2.20)$$

This solution is again unstable for any physical values of the parameters: the sum is real only for $M^2 < 1/4$ which is outside the required range.

Another generalization is to the case of the 3-spin circular constant-radii solutions of [7] belonging to the $SU(3)$ sector. In particular, for the case of the solution with $J_1 = J_2,\ J_3 \neq 0,\ J = 2J_1 + J_3$ (related by an $SO(6)$ rotation to the solution of [5, 37] but again having manifestly integer-labelled fermionic fluctuation modes) one readily finds the analogs of (2.19) and (2.20) using the expressions in [37, 38]. This solution is stable for large enough J_3, so the 1-loop correction to the energy is real. The classical energy is

$$E_0 = J + \frac{\lambda}{2J} k^2 s^2 + O(\frac{\lambda^2}{J^3})\ , \qquad s^2 \equiv 1 - \frac{J_3}{J}\ . \qquad (2.21)$$

Expanding the fluctuation frequencies given in [37, 38] in $\frac{1}{\mathcal{J}^2} = \frac{\lambda}{J^2}$ one observes again that contributions of other superstring modes combine to make the sum of the corresponding Landau-Lifshitz frequencies finite (the latter were identified [19] directly in the $SU(3)$ Landau-Lifshitz model on CP^2 [19, 20] and reproduced also on the $SU(3)$ spin chain side in [42])[11]

$$E^{(1)}_{\rm zero} = E_{\rm reg} = \frac{\lambda}{2J^2}[\sum_{n=1}^{\infty}(-2n^2 + 2k^2 s^2)]_{\rm reg} = \frac{\lambda}{2J^2} k^2 s^2\ , \qquad (2.22)$$

[11] The contributions of the two Landau-Lifshitz frequencies may be combined as in (2.4) together using the identity $\sqrt{a-b} + \sqrt{a+b} = \sqrt{2a^2 + 2\sqrt{a^2 - b^2}}$. This solution is stable for $s^2 \le 1 - (1 - \frac{1}{2k})^2$. To leading order in $1/\mathcal{J}$ the parameter s^2 is the same as the parameter $q = \sin^2\gamma_0$ used in [37].

$$
\begin{aligned}
E^{(1)}_{\rm non-zero} = E_{\rm fin} &= \frac{\lambda}{2J^2}\sum_{n=1}^{\infty}\Big[n\sqrt{n^2+2(2-3s^2)k^2+2k\sqrt{4n^2(1-s^2)-k^2s^2(8-9s^2)}} \\
&+ n\sqrt{n^2+2(2-3s^2)k^2-2k\sqrt{4n^2(1-s^2)-k^2s^2(8-9s^2)}} \\
&- 2n^2+2k^2s^2\Big] \ . \qquad (2.23)
\end{aligned}
$$

This reduces back to (2.5),(2.6) or (2.14),(2.15) for $J_3=0$, i.e. $s=1$[12]. We see again that zero-mode contribution is the same as the ζ-function regularized singular part of the sum of the Landau-Lifshitz fluctuation mode contributions, and that again $E_{\rm reg}$ is simply the classical term in (2.21) times $1/J$. This suggests that it should come out of the non-anomalous finite-size correction in the $SU(3)$ spin chain generalization of the $SU(2)$ computation in [40].[13]

Essentially the same conclusions were reached in [39] in the case of a rigid circular solution in the $SL(2)$ sector [7] carrying spin S in AdS_5 and spin J in S^5. This solution may be viewed as a "naive" analytic continuation of the above (J_1,J_2) solution in the $SU(2)$ sector.[14] The non-zero of 3+3 $AdS_5\times S^5$ complex embedding coordinates are $\mathrm{Y}_0=\cosh\rho_0\ e^{i\kappa\tau}$, $\mathrm{Y}_1=\sinh\rho_0\ e^{iw\tau+im\sigma}$, $\mathrm{X}_1=e^{i\omega\tau-ik\sigma}$.[15] The classical energy is

$$
E = J+S+\frac{\lambda}{2J}M^2+O(\frac{\lambda^2}{J^3}) \ , \qquad (2.24)
$$

$$
M^2\equiv m^2\alpha(1+\alpha) \ , \qquad \alpha\equiv\frac{S}{J} \ , \qquad mS=kJ \ . \qquad (2.25)
$$

The expression for the 1-loop string correction to te energy $E_1=E_{\rm zero}+E_{\rm non-zero}$ which is again the same as the regularized sum of the $SL(2)$ Landau-Lifshitz fluctuation mode contributions are the analogs of (2.5),(2.16), (2.6),(2.15) which happen to be simply (2.19),(2.20) with $M^2\to -M^2$ [39][16]

$$
E_{\rm zero}=E_{\rm reg}+O(\frac{\lambda^2}{J^4}) \ , \qquad E_{\rm reg}=\frac{\lambda}{2J^2}[\sum_{n=1}^{\infty}(n^2+2M^2)]_{\rm reg}=-\frac{\lambda}{2J^2}M^2 \ , \qquad (2.26)
$$

[12]The above expression can be readily generalized to the case of the generic circular solution of [7] with three unequal spins J_i. Computation of the 1-loop string effective action near such general solution was recently discussed by H. Fuji and Y. Satoh (to appear).

[13]This was indeed confirmed while this paper was in preparation in [43].

[14]Direct analytic continuation in the spirit of [9] ($(E,S,J)\to(-J_1,J_2,-E)$, etc.) leads to a problem of periodic time coordinate and thus should be supplemented by an additional redefinition of τ and σ. Most of the leading-order relations for the energy and the fluctuation frequencies are still very similar, cf. [7, 39].

[15]Here we interchange the notation for the AdS_5 and S^5 winding numbers $m\leftrightarrow k$ compared to [39] (we choose $m,k>0$). We also use α instead of u in [39] as a notation for the spin ratio $\frac{S}{J}$.

[16]We take into account a slight correction to the expression in the original version of [39] (to be done in its revised version) removing the splitting the sum over n into two parts.

$$E_{\rm non-zero} = E_{\rm fin} + O(\frac{\lambda^2}{J^4}) \ , \qquad E_{\rm fin} = \frac{\lambda}{2J^2}\sum_{n=1}^{\infty}(n\sqrt{n^2+4M^2} - n^2 - 2M^2) \ . \quad (2.27)$$

In contrast to its $SU(2)$ analog, this (S, J) solution is always stable, so that E_1 is real and can be directly compared to the $SL(2)$ spin chain result for the finite-size correction (that the classical term in (2.24) matches the leading Bethe ansatz result was already shown in [26]).

This is what we are going to do in the next section.

3 Bethe ansatz: finite size corrections

3.1 The $SL(2)$ sector

We shall first consider the $SL(2)$ sector which is not plagued by instabilities and for that reason is conceptually simpler. The Bethe ansatz for $SL(2)$ is also technically simpler because all Bethe roots are real. The $SL(2)$ sector consists of operators of the form $\operatorname{tr} D_+^S Z^J$. These operators are dual to strings with the spin S in AdS_5 and the angular momentum J in S^5. The spectrum of anomalous dimensions of the $SL(2)$ operators is described at one loop by solutions of the Bethe equations [44]:

$$\left(\frac{u_k - i/2}{u_k + i/2}\right)^J = \prod_{j\neq k}^{S} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \qquad (3.1)$$

where $k, j = 1, \ldots, S$ and J plays the role of the spin chain length. The one-loop anomalous dimension is

$$E - S - J = \frac{\lambda}{8\pi^2 J^2}\sum_{k=1}^{S}\frac{1}{u_k^2 + 1/4}\,. \qquad (3.2)$$

The cyclicity of the trace in the SYM operators – equivalent to the translational invariance of the wave function – imposes an additional constraint:

$$\prod_{k=1}^{S}\frac{u_k - i/2}{u_k + i/2} = 1. \qquad (3.3)$$

It is useful to keep in mind that solutions of the Bethe equations that do not satisfy this condition still correspond to perfectly well-defined eigenstates of the underlying spin chain. They carry a non-zero total momentum and thus have no interpretation in the SYM theory.

In order to take the large-J limit we rescale the Bethe roots as $u_k = Jx_k$ (x_k remain finite at $J \to \infty$), take the logarithm of both sides of the Bethe equation and expand in $1/J$:

$$2\pi m - \frac{1}{x_k} = \frac{1}{i}\sum_{j\neq k}^{S}\ln\frac{x_k - x_j + i/J}{x_k - x_j - i/J}\,. \qquad (3.4)$$

The omitted terms are of order $O(1/J^2)$ and therefore affect neither the leading order nor the $1/J$ correction. An arbitrary phase $2\pi m$ arises because of the arbitrariness in choosing the branch of the logarithm and, in principle, could be different for different roots. Requiring that the phase is the same for all roots is a strong restriction and singles out a particular class of states [26]. These states are dual to those string solutions whose fluctuation spectrum is discussed in the previous sections. Let us define the branch of the logarithm in (3.4) with the help of the integral representation:

$$2\pi m - \frac{1}{x_k} = 2\sum_{j\neq k}\int_0^{1/J} d\varepsilon\, \frac{x_k - x_j}{(x_k - x_j)^2 + \varepsilon^2}\,. \tag{3.5}$$

This completely fixes the ambiguity in the definition of the mode number m.

Formally, the logarithm on the right hand side of (3.4) can be also expanded in $1/J$. The expansion is accurate for most of the Bethe roots, because normally $x_k - x_j \sim x_k \sim O(1)$, but for a small fraction of nearby roots with $|k-j| \ll J$ the expansion breaks down since then $x_k - x_j$ is of order $1/J$. The local contribution produces an *anomaly* which affects the $O(1/J)$ corrections and which has been overlooked in the previous analyses. We shall calculate the anomaly by extending the approach to finite-size corrections developed in [40].

In order to solve the Bethe equations in the thermodynamic limit we introduce the resolvent

$$G(x) = \frac{1}{J}\sum_{k=1}^{S}\frac{1}{x - x_k}\,. \tag{3.6}$$

It has the following asymptotics at infinity:

$$G(x) = \frac{\alpha}{x} + \ldots \qquad (x \to \infty), \tag{3.7}$$

where

$$\alpha = \frac{S}{J}\,. \tag{3.8}$$

The total energy and the total momentum are Taylor coefficients of $G(x)$ at zero:

$$P = -G(0), \qquad E - S - J = -\frac{\lambda}{8\pi^2 J}\, G'(0). \tag{3.9}$$

The momentum condition (3.3) requires $G(0)$ to be an integer multiple of 2π.

The Bethe equations can be written in the scaling limit entirely in terms of the resolvent. The derivation proceeds as follows. Let us multiply both sides of (3.5) by $1/(x - x_k)$ and sum over k. Observing that

$$\sum_{j\neq k}\frac{1}{x - x_k}\,\frac{x_k - x_j}{(x_k - x_j)^2 + \varepsilon^2} = \frac{J^2}{2}\,G^2(x) + \frac{J}{2}\,G'(x) - \frac{1}{2}\sum_{j\neq k}\frac{1}{x - x_k}\,\frac{1}{x - x_j}\,\frac{\varepsilon^2}{(x_k - x_j)^2 + \varepsilon^2}\,, \tag{3.10}$$

we find:

$$G^2(x) - \left(2\pi m - \frac{1}{x}\right) G(x) + \frac{G(0)}{x} = \frac{1}{J}\left[\sum_{j\neq k} \frac{1}{x-x_k}\,\frac{1}{x-x_j}\int_0^{1/J} d\varepsilon\,\frac{\varepsilon^2}{(x_k-x_j)^2+\varepsilon^2} - G'(x)\right]. \tag{3.11}$$

Let us now take $J \to \infty$. Then only $x_k - x_j \sim 1/J$ contribute to the sum on the right hand side. Hence, the sum is dominated by the local distribution of Bethe roots, which is approximately linear:

$$x_k - x_j \approx \frac{k-j}{J\rho(x_k)}\,, \tag{3.12}$$

where $\rho(x)$ is the macroscopic density:

$$\rho(x) = \frac{1}{J}\sum_{k=1}^{S}\delta(x-x_k) = \frac{1}{2\pi i}\left(G(x+i0) - G(x-i0)\right). \tag{3.13}$$

Thus,

$$\begin{aligned}\sum_{j\neq k}\frac{1}{x-x_k}\,\frac{1}{x-x_j}\,\frac{\varepsilon^2}{(x_k-x_j)^2+\varepsilon^2} \quad &\approx \quad \frac{1}{(x-x_k)^2}\sum_{n\neq 0}\frac{\varepsilon^2 J^2\rho^2(x_k)}{n^2+\varepsilon^2 J^2\rho^2(x_k)} \\ &= \quad \frac{\pi\varepsilon J\rho(x_k)\coth(\pi\varepsilon J\rho(x_k)) - 1}{(x-x_k)^2}\,,\end{aligned} \tag{3.14}$$

and finally we get:

$$G^2(x) - \left(2\pi m - \frac{1}{x}\right) G(x) + \frac{G(0)}{x} = \frac{1}{J}\int \frac{dy\,\tilde{\rho}(y)}{(x-y)^2}\,, \tag{3.15}$$

where

$$\tilde{\rho} = \frac{1}{\pi}\int_0^{\pi\rho} d\xi\ \xi\ \coth\xi\ . \tag{3.16}$$

The equation (3.15) can be solved perturbatively in $1/J$:

$$\begin{aligned}G(x) = \pi m - \frac{1}{2x} \quad &- \quad \frac{\sqrt{(2\pi m x - 1)^2 - 8\pi m\alpha x}}{2x} \\ &- \quad \frac{1}{J}\,\frac{x}{\sqrt{(2\pi m x-1)^2 - 8\pi m\alpha x}}\int\frac{dy\,\tilde{\rho}(y)}{(x-y)^2}\,.\end{aligned} \tag{3.17}$$

The momentum condition, $G(0) = -2\pi k$, requires $m\alpha = k$ to be an integer so that $mS = kJ$. This is the same as in string theory, but we can consider states with any

α as far as the spectrum of the spin chain is concerned. However, the solutions with irrational α do not correspond to any operators in $\mathcal{N}=4$ SYM.

Using (3.9) we find for the order λ terms in the energy (3.2)

$$E_0 = \frac{\lambda m^2 \alpha(1+\alpha)}{2J}, \tag{3.18}$$

$$E_1 = -\frac{\lambda}{8\pi^2 J^2}\int \frac{dx\,\tilde{\rho}(x)}{x^2}, \tag{3.19}$$

where the effective density $\tilde{\rho}$ is defined in (3.16). The true density is given by

$$\rho(x) = \frac{\sqrt{8\pi m\alpha x - (2\pi m x - 1)^2}}{2\pi x}. \tag{3.20}$$

Interchanging the order of integrations in (3.19), (3.16) and rescaling the integration variable we get:

$$E_1 = -\frac{2\lambda M^3}{J^2}\int_{-1}^{1} dx\, x\sqrt{1-x^2}\,\coth\left(2\pi M\,x\right), \tag{3.21}$$

$$M \equiv m\sqrt{\alpha(1+\alpha)}\,. \tag{3.22}$$

This is our final result.

The integral in (3.21) cannot be expressed in elementary functions, but we can easily find its asymptotics at small and large filling fraction α:

$$E_1 = -\frac{\lambda m^2}{2J^2}\left[\alpha + \left(1+\frac{\pi^2 m^2}{3}\right)\alpha^2 + \ldots\right] \qquad (\alpha \to 0), \tag{3.23}$$

$$E_1 = -\frac{4\lambda m^3 \alpha^3}{3J^2} + \ldots \qquad (\alpha \to \infty). \tag{3.24}$$

To compare to the string theory result let us convert the finite integral in (3.21) into a sum using the identity [17]

$$\pi a \coth(\pi a) = a^2 \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = 1 + 2a^2 \sum_{n=1}^{\infty}\frac{1}{n^2+a^2}\,, \tag{3.25}$$

with $a = 2Mx$ in (3.21). Then doing the integral over x first we reproduce the string-theory result (2.26),(2.27) of [39], i.e.

$$E_1 = -\frac{\lambda M^2}{2J^2} + \frac{\lambda}{2J^2}\sum_{n=1}^{\infty}\left(n\sqrt{n^2+4M^2} - n^2 - 2M^2\right) + O(\frac{\lambda^2}{J^4})\,, \tag{3.26}$$

where the first term is the contribution of the zero modes.[18]

[17] This is equivalent to undoing the summation in (3.14). It is interesting that the mode numbers of the frequencies (n's) in the Bethe ansatz have the meaning of the distances between the roots along the contour ($k-j$ in (3.12)). We have no explanation for this fact.

[18] It is possible to do the same calculation in a different way. First we write the integral over x

3.2 The $SU(2)$ sector and instabilities

The Bethe equations for the $SU(2)$ operators $\mathrm{tr}(Z^{J_1}W^{J_2} + \text{permutations})$ [32] differ from their $SL(2)$ counterpart (3.1) by reversing the signs on the left hand side:

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^J = \prod_{j\neq k}^{J_2} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \tag{3.27}$$

where $J = J_1 + J_2$. All the previous calculations for the $SL(2)$ case can be literally repeated in this case. The only subtle point is eq. (3.12), because the roots are now complex. But the density is also complex – it is the product $dx\,\rho(x)$ that must be real and positive – so (3.12) holds even if the roots lie on the complex plane. In fact, we need not do separate calculations for the $SU(2)$ case since the one-loop anomalous dimensions in the the $SU(2)$ and $SL(2)$ sectors are related by the analytic continuation in the filling fraction. This fact was established for the thermodynamic limit in [9] and holds true for the leading $1/J$ correction as well. If we define

$$\alpha = \frac{J_2}{J} = \frac{J_2}{J_1 + J_2}\,, \tag{3.28}$$

then the finite-size correction in the $SU(2)$ case can be obtained from (3.21) by the substitution $\alpha \to -\alpha$:

$$E_1 = \frac{2\lambda M^3}{J^2}\int_{-1}^{1} dx\, x\sqrt{1-x^2}\,\cot\left(2\pi M\, x\right), \tag{3.29}$$

$$M \equiv m\sqrt{\alpha(1-\alpha)}\,. \tag{3.30}$$

This formula makes sense only if M is sufficiently small, and then all the poles of the integrand lie outside of the region of integration. If we analytically continue E_1 past

$$\alpha_c = \frac{m - \sqrt{m^2-1}}{2m}\,, \tag{3.31}$$

the poles hit the contour of integration and E_1 acquires an imaginary part from the residue. The poles are associated with the frequencies of fluctuations around the corresponding solution of the Landau-Lifshitz equation, and the imaginary frequency signals that the solution becomes unstable. Note that the momentum condition requires integrality of $m\alpha = k$ and implies that $M \geq 1$, so the string states dual to the SYM operators are always unstable, in accord with the analysis of [5, 7].

as a contour integral around the cut of the square root and then deform the contour to encircle the poles of $\coth(2\pi M x)$ at $x = \pi n i/2M$. This leads to a divergent series because the integrand does not fall sufficiently fast at infinity and therefore it is necessary to do subtractions before deforming the contour. Subtracting and adding $x^2 - 1/2$ from $x\sqrt{x^2-1}$ gives the finite sum over the non-zero modes, the subtraction can be evaluated by shrinking the contour to zero where coth has a pole and this produces a zero-mode contribution.

How do we see this instability in the Bethe equations? It is easy to understand what goes wrong. The density of Bethe roots grows with α and at $\alpha = \alpha_c$ some roots appear to be separated by i/J. This causes problems in the calculation of the anomaly – the sum in (3.14) becomes ill-defined since the denominator of the $n = 1$ term turns to zero. In fact, the solutions with $\alpha > \alpha_c$ violate the basic assumption used in deriving the macroscopic equations for the distribution of Bethe roots, namely the assumption that all logarithms $\ln(x_k - x_j + i/J)/(x_k - x_j - i/J)$ belong to the same branch. At the leading order only well-separated roots are important. The argument of the ln is close to one for them and it is easy to forget about this assumption. Indeed, the derivation of scaling solutions of the Bethe equations, which are dual to the semiclassical string states [8, 9, 25], starts with rewriting the microscopic equations in the logarithmic form:

$$\sum_{j \neq k} \ln \frac{x_k - x_j + i/J}{x_k - x_j - i/J} = 2\pi i m_k + J \ln \frac{x_k + i/2J}{x_k - i/2J}\,, \tag{3.32}$$

where $x_k = u_k/J$. Then these equations are expanded in $1/J$ and rewritten as an integral equation for the density of roots:

$$2 \fint \frac{dy\, \rho(y)}{x - y} = 2\pi m_I + \frac{1}{x}\,, \qquad x \in C_I. \tag{3.33}$$

The density is supported on a set of contours C_I in the complex plane. The integral equation can be now solved and its general solution can be expressed in terms of hyperelliptic integrals [25]. This derivation tacitly assumes that the logarithms in (3.32) are single-valued and that all roots with the same phase lie on the same contour C_I. While this assumption is certainly justified for well-separated roots, it can break down if $x_k - x_j = O(1/J)$. For such x_k and x_j the local approximation (3.12) is accurate and the logarithm in (3.32) takes the form

$$\ln \frac{n + i\rho(x)}{n - i\rho(x)} \equiv F_n(x), \tag{3.34}$$

where $n = k - j$. We should require that $F_n(x)$ is single-valued along each C_I. Otherwise the phase $2\pi m_I$ will jump by an integer multiple of 2π somewhere in the middle of the contour. This extra condition does not follow from the macroscopic equations (3.33) themselves and should be imposed by hand. In effect, only those solutions of the classical Bethe equation correspond to microscopic Bethe states which satisfy the following

Stability condition: *The density of roots must satisfy*

$$\Delta_{C_I} \arg \frac{n + i\rho(x)}{n - i\rho(x)} = 0 \tag{3.35}$$

for any integer n and for all contours C_I. The density is zero at the ends of all cuts, so $(n + i\rho(x))/(n - i\rho(x))$ traverses a closed curve in the complex plane which begins and ends in 1. The solution is stable if this curve does not encircle the origin.

For many interesting solutions there is only one point ($x = x_*$) at which dx, and thus $\rho(x_*)$, is pure imaginary and consequently the curve $(n+i\rho(x))/(n-i\rho(x))$ crosses the real axis only once, at $x = x_*$. In that case the solution is stable iff the crossing point lies to the right of the origin, or if

$$|\rho(x_*)| < 1. \tag{3.36}$$

This condition has a simple meaning: when the difference between adjacent roots is pure imaginary, it should be smaller than i/J. This makes the logarithm in (3.4) single-valued.

We conjecture that solutions of the classical Bethe equation that violate the stability condition correspond to unstable solutions of the Landau-Lifshitz equation. Consider, for example, the rational (single-cut) solution:

$$\rho(x) = \frac{i\sqrt{8\pi m\alpha x + (2\pi m x - 1)^2}}{2\pi x}\,. \tag{3.37}$$

The density is supported on a single contour which crosses the real axis at

$$x_* = \frac{1}{2\pi m(1-2\alpha)}\,, \tag{3.38}$$

and

$$\rho(x_*) = 2mi\alpha\sqrt{1-\alpha}\,. \tag{3.39}$$

The stability condition (3.36) demands

$$\alpha < \alpha_c, \tag{3.40}$$

where α_c is defined in (3.31), which is precisely the condition for linear stability of the corresponding classical solution.

The stability condition is very similar to the consistency condition for the Douglas-Kazakov solution of large-N QCD on a two-dimensional sphere [45], where the density is also bounded from above. When the bound is saturated, the solution undergoes a phase transition and develops a patch with flat distribution. Similar phase transition will occur here. If the stability bound gets violated on a contour C_I, for example, if the filling fraction of the rational solution (3.37) exceeds the critical value α_c, roots on different parts of C_I will have different mode numbers m_K and therefore the contour will break in two pieces on which the mode numbers are constant. For instance, the rational solution (3.37) will become two-cut. The two cuts will be connected by a condensate in which Bethe roots are exactly equidistant: $x_{k+1} - x_k = i/J$. The solutions with such condensates are discussed in more detail in [8, 9, 25].

Finally, let us write down the $1/J$ correction to the classical Bethe equation (3.33):

$$2\fint \frac{dy\,\rho(y)}{x-y} = 2\pi m_I + \frac{1}{x} - \frac{1}{J}\,\pi\rho'(x)\coth\pi\rho(x), \qquad x \in C_I. \tag{3.41}$$

One derivation is given in Appendix A. The other can proceed by deriving an analog of eq. (3.15) for arbitrary mode numbers and then taking its discontinuity across the cuts C_I. We only know how to solve for the $1/J$ corrections in the simplest case of the rational solutions, but it is conceivable that the finite-size corrections can be computed in the same generality in which the leading-order solution is known [25].

Perhaps (3.41) can be solved by iterations.[19] The leading order is known. Solving for the next iteration reduces to inverting the Hilbert kernel, which can be done by fairly standard techniques [46].

4 Concluding remarks

It has been observed by direct computations that the energies of the semiclassical strings agree with the anomalous dimensions at two gauge-theory loops and start to disagree at three loops. This is true for classical macroscopic strings, as well for short strings in the near-BMN limit. We believe that quantum string/finite-size corrections should be no exception and that they should agree at two loops. The two-loop calculation on the gauge-theory side is certainly possible, since the two-loop Bethe ansatz for the $SL(2)$ sector is known [31]. In fact, the two-loop corrections do not affect the anomaly and only change the macroscopic part of the classical Bethe equation. The expansion of the string one-loop quantum correction to the "gauge" two-loop order $O(\lambda^2/J^4)$ meets certain technical difficulties. The sum of the frequencies expanded in λ/J^2 diverges at $O(\lambda^2/J^4)$ and it is necessary to first resum the series and then expand. One can of course resort to numerical evaluation of the sum.

Another obvious extension of our results is the calculation of the $1/J$ corrections from the quantum string Bethe ansatz [35, 31], which was conjectured to describe the string spectrum at strong coupling but beyond the semiclassical limit. Comparison of the $1/J$ corrections computed from the Bethe ansatz with the explicit string calculation would be a strong test of the conjecture of [35, 31].

We also believe that our approach can be generalized to the $SU(3)$ sector. While this paper was in preparation, there appeared an interesting work [43] that generalized the computation of non-anomalous part of finite-size correction in [40] from $SU(2)$ to $SU(3)$ case. As we have explained above, this non-anomalous "$1 \to 1 + 1/J$" correction to the classical energy corresponds to the zero-mode string contribution and should again be supplemented by the anomaly, after which we expect the spin-chain result to agree with the string-theory prediction, i.e. the sum of (2.22) and (2.23).

Potentially, $1/J$ corrections can be also computed for many other known solutions for classical strings in $AdS_5 \times S^5$, e.g., for folded $SU(2)$ string solution of [47] dual to the two-cut solution of the Bethe ansatz [8].

An interesting open problem is to show that in general the finite-size correction is

[19] We would like to thank I. Kostov for the discussion of this point.

always given by a sum over energies (obtained by removing one root from a cut [8]) of the nearby fluctuation modes in a given spin-chain sector, i.e. by the (regularized) sum of the energies of the Landau-Lifshitz modes.

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A Anomaly

In this appendix we demonstrate the effect of the anomaly on the Bethe equations for Bethe strings in the thermodynamic limit. In the leading order in $1/J$ the anomaly the anomaly is absent [49], but it does contribute to subleading orders: A Bethe string is a collection of Bethe roots u_n distributed on a curve C in the complex plane. Let us focus on a particular point of this curve, for convenience we shall assign the index $n = 0$ to this point. Then for large J we can expand the positions of Bethe roots close to u_0 as follows

$$u_n = aJ + bn + \frac{1}{2}cn^2/J + \dots \tag{A.1}$$

where $a, b, c, \dots$ depend on J but are finite for $J \to \infty$. The scattering phase for u_0 is

$$\phi = \frac{1}{i}\sum_n \log\frac{u_0 - u_n + i}{u_0 - u_n - i} \tag{A.2}$$

Substituting the above u_n into ϕ and separating into small and large n we get

$$\phi = \frac{1}{i}\sum_{|n|<N} \log\frac{-bn - cn^2/2J + i}{-bn - cn^2/2J - i} + \sum_{|n|>N}\frac{2}{u_0 - u_n} + \dots \tag{A.3}$$

Then let us sum up terms with positive and negative n in the first term and expand

$$\phi = \sum_{n=1}^{N} \left(\frac{2c/J}{b^2} - \frac{2c/J}{b^2(1+b^2n^2)} \right) + \sum_{|n|>N} \frac{2}{u_0 - u_n} + \dots \tag{A.4}$$

The first term can be absorbed into the last one because $\frac{2}{u_0-u_n} = 2c/Jb^2$ for small n. The sum in the second term converges linearly, thus we can send $N \to \infty$ at this order,

$$\phi = \sum_n \frac{2}{u_0 - u_n} + \frac{c}{b^2 J}(1 - \frac{\pi}{b}\coth(\frac{\pi}{b})) + \dots \tag{A.5}$$

The second term is the anomaly. The hyperbolic cotangent represents the effect of the poles of the nearby Bethe roots. It can be computed using the leading order density $\rho(x) = dn/dx$ with $xJ = u$. Then $b = 1/\rho(x_0)$ and $c = -\rho'(x_0)/(\rho(x_0))^3$ and thus the anomaly contribution is

$$\delta\phi = \frac{1}{J}\frac{\rho'(x_0)}{\rho(x_0)}(\pi\rho(x_0)\coth(\pi\rho(x_0)) - 1) \tag{A.6}$$

We see that it is a purely local term. Finally, we can approximate the sum by an integral

$$\phi = ⨍ \frac{2\,dx\,\rho(x)}{x_0 - x} + \frac{1}{J}\pi\rho'(x_0)\coth(\pi\rho(x_0)) + \dots \tag{A.7}$$

The term $\rho'(x_0)/\rho(x_0)$ was absorbed by turning the sum into an integral.

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